Lecture 10 - Moment of Inertia

A Puzzle...

<u>Question</u>

For any object, there are typically many ways to calculate the moment of inertia $I = \int r^2 \, dm$, usually by doing the integration by considering different layers or different coordinates. What are some ways to compute this integral for a solid sphere?

<u>Solution</u>

Denote the mass density $\rho = \frac{M}{\frac{4}{3}\pi R^3}$. We will use *Mathematica* to perform the integrations.

1. The simplest and most intuitive way is to perform the integration using spherical coordinates. The distance from the z-axis equals $r \operatorname{Sin}[\theta]$ and the volume element is $r^2 \operatorname{Sin}[\theta] \, dr \, d\theta \, d\phi$,

$$I = \int_0^{2\pi} \int_0^{\pi} \int_0^R (r \operatorname{Sin}[\theta])^2 \rho \, r^2 \operatorname{Sin}[\theta] \, dr \, d\theta \, d\phi \tag{1}$$

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Integrate[(r*Sin[θ])^2 ρ r^2 Sin[θ], {r, 0, R}, {θ, 0, π}, {ϕ, 0, 2 π}] /. ρ → M/(4/3 π R^3)
```

$$\frac{2 M R^2}{5}$$

2. This integral is more complex in Cartesian coordinates, but still straightforward. The distance from the z-axis is $(x^2 + y^2)^{1/2}$ and the volume element is $dx \, dy \, dz$,

$$I = \int_{-R}^{R} \int_{-(R^2-z^2)^{1/2}}^{(R^2-z^2)^{1/2}} \int_{-(R^2-y^2-z^2)^{1/2}}^{(R^2-y^2-z^2)^{1/2}} (x^2 + y^2) \, \rho \, dx \, dy \, dz \tag{2}$$

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Integrate[(x^2 + y^2) ρ, {z, -R, R}, {y, -Sqrt[R^2 - z^2], Sqrt[R^2 - z^2]}, {x, -Sqrt[R^2 - y^2 - z^2], Sqrt[R^2 - y^2 - z^2]}] /. ρ → M/(4/3 π R^3)
```

$$\frac{2 M R^2}{5}$$

3. We could integrate up along concentric hollow spheres. The moment of inertia of a hollow sphere of mass M and radius R through any axis passing through its origin equals $\frac{2}{3} M R^2$ (as an exercise, prove this to yourself). Therefore, the mass of the hollow sphere of radius r and thickness dr equals $\rho \, 4 \pi r^2 \, dr$. Thus, the moment of inertia of a solid sphere equals

$$I = \int_0^R \frac{2}{3} (\rho \, 4 \pi r^2) \, r^2 \, dr \tag{3}$$

```
Integrate[2/3*(ρ 4 π r^2)*r^2, {r, 0, R}] /. ρ → M/(4/3 π R^3)
```

$$\frac{2 M R^2}{5}$$

4. Getting a bit fancier, we can break up the integration along disks centered on the z-axis. As we will find out in today's lecture, the moment of inertia of a disk (or cylinder) with mass M and radius R equals $\frac{1}{2} M R^2$. Defining

$r[z]$ and $m[z]$ to be the radius and mass of the thin disk centered at (0, 0, z) with width dz,

$$I = \int \frac{1}{2} m[z]\, r[z]^2\, dz \qquad (4)$$

where $r[z] = \left(R^2 - z^2\right)^{1/2}$ and $m[z]\, dz = \rho\, \pi\, r[z]^2\, dz$.

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In[2]:= Integrate[1/2 (ρ π (R^2 - z^2)) (R^2 - z^2), {z, -R, R}] /. ρ → M/(4/3 π R^3)

Out[2]= 2 M R^2/5
```

5. You could go on imagining other crazy ways to do this integration. For example, you could integrate along disks centered on the x-axis (this is actually trivial once we learn about the Parallel Axis Theorem and the Perpendicular Axis Theorem in today's lecture). I encourage you to try this route yourself!

Moment of Inertia

In the last lecture, we analyzed the angular momentum of extended objects and saw that a new quantity, the moment of inertia, appeared in the resulting equations. In this lecture, we will understand what the moment of inertia is by calculating it for various objects. That will prepare us to discuss rotational dynamics in the coming weeks.

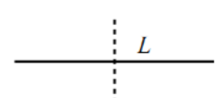
Recall that the moment of inertia is calculated about *an axis*, so in each problem we specify around what axis we are rotating our object of interest. This is very different from torque and angular momentum, which are calculated about *a point*.

Supplemental Section: Finding the Center of Mass

Supplemental Section: Moment of Inertia in 1D

Example

Calculate the moment of inertia of a thin rod of mass M and length L about an axis perpendicular to the center of the rod.



Solution

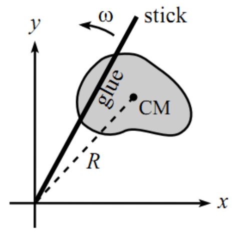
Defining the mass density $\rho = \frac{M}{L}$,

$$I = \int r^2\, dm = \int_{-L/2}^{L/2} x^2 \rho\, dx = \frac{1}{3}\left\{\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right\}\rho = \frac{1}{3}\left\{\frac{1}{4} L^3\right\}\rho = \frac{1}{12} M L^2 \qquad (10)$$

This useful result is definitely worth knowing, as it will often appear when computing moments of inertia of various objects (especially in limiting cases)! □

Supplemental Section: Parallel Axis Theorem

Consider an object whose center of mass rotates around the origin at the same rate as the body rotates around the center of mass. This may be achieved, for example, by gluing a stick across the pancake and pivoting one end of the stick at the origin



In this special case, all points in the object travel in circles around the origin. Let their angular velocity be ω. Breaking up the angular momentum into the translational portion (as if the entire object's mass was concentrated at its center of mass) and a rotational portion (from the object spinning about its center of mass with angular velocity ω),

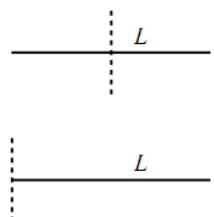
$$L_z = (M R^2 + I_{\text{CM}})\, \omega \tag{11}$$

In other words, the moment of inertia about the origin equals

$$I = M R^2 + I_{\text{CM}} \tag{12}$$

Note that this relation is only true with the distance R measured relative to the center of mass (it does not work for two general points).

<u>Example</u>
Consider a thin rod of mass M and length L. Given that the moment of inertia through an axis passing through its center and perpendicular to the rod equals $I_{\text{CM}} = \frac{1}{12} M L^2$, what is the moment of inertia I_{end} through an axis passing through an axis passing through the end of the rod and perpendicular to the rod?

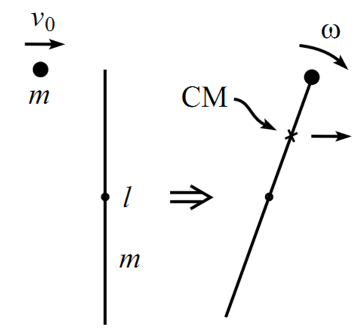


<u>Solution</u>
Using the parallel axis theorem,

$$I_{\text{end}} = M\left(\frac{L}{2}\right)^2 + I_{\text{CM}} = \frac{1}{4} M L^2 + \frac{1}{12} M L^2 = \frac{1}{3} M L^2 \tag{13}$$

Remember that this works only with the center of mass. If we instead want to compare I_{end} with the I around a point $\frac{L}{6}$ from that end, then we cannot say that they differ by $M\left(\frac{L}{6}\right)^2$. But we can compare each of them to I^{CM} and say that they differ by $M\left(\frac{L}{2}\right)^2 - M\left(\frac{L}{3}\right)^2 = \frac{5}{36} M L^2$. □

<u>Example</u>
In the last lecture we considered a mass m coming in and sticking to the end of a thin rod of mass m and length l. This final object will rotate about its center of mass. What is its I_{CM}?



Solution
Note that the center of mass of the final object is $\frac{l}{4}$ away from its center. I_{CM} will have a contribution from the thin rod and from the mass m. The former can be calculated using the parallel axis theorem

$$I_{CM}^{rod} = \frac{1}{12} m l^2 + m \left(\frac{l}{4}\right)^2 = \frac{7}{48} m l^2 \tag{14}$$

$$I_{CM}^{mass} = m \left(\frac{l}{4}\right)^2 = \frac{1}{16} m l^2 \tag{15}$$

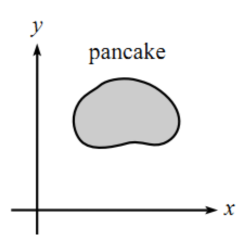
Now I_{CM} can be calculated by adding these two contributions

$$I_{CM} = I_{CM}^{rod} + I_{CM}^{mass} = \frac{5}{24} m l^2 \tag{16}$$

which we could have substituted into this previous problem. □

Supplemental Section: Perpendicular Axis Theorem

This theorem is valid *only* for pancake objects. Consider a pancake object in the x-y plane.



If the object were spinning with an angular velocity ω about the z-axis, we know that its moment of inertia about the z-axis equals

$$I_z = \int r^2 \, dm = \int (x^2 + y^2) \, dm \tag{17}$$

Imagine that this object was spinning about the x-axis with angular velocity ω. Then we define the moment of inertia I_x about the x-axis as $I_x = \frac{L_x}{\omega}$ which would have the form

$$I_x = \int (y^2 + z^2) \, dm \tag{18}$$

Similarly, we define the moment of inertia I_y about the y-axis by imagining this object spinning about the y-axis with angular velocity ω,

$$I_y = \int (z^2 + x^2) \, dm \tag{19}$$

For a pancake object, $z = 0$ everywhere along the object, and therefore we find

$$I_z = I_x + I_y \tag{20}$$

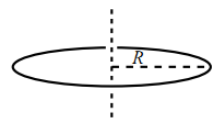
Although this does not apply frequently, when it does it can help save some computation.

Supplemental Section: Calculating Moments of Inertia

Calculating the moment of inertia for an object is an extremely useful skill, since it not only uses your math knowledge but also provides excellent preparation for the types of integrals you will do in Electrodynamics!

Example

Calculate the moment of inertia of a ring of mass M and radius R about the axis going through the ring's center perpendicular to the plane of the ring.



Solution

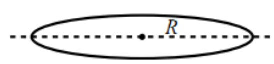
Orient the ring in the x-y plane so that we calculate the moment of inertia about the z-axis. Using the mass density $\rho = \frac{M}{2\pi R}$, the moment of inertia I_z about this axis equals

$$I_z = \int r^2\, dm = \int_0^{2\pi} R^2\, \rho\, R\, d\theta = (2\pi R \rho) R^2 = M R^2 \tag{21}$$

as expected, since the entire mass M of the object is at a distance R from the axis. □

Example

Calculate the moment of inertia of a ring of mass M and radius R about the axis going through the ring's center parallel to the plane of the ring.



Solution

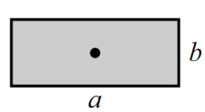
Orient the ring in the x-y plane so that we calculate the moment of inertia about the x-axis.

$$I_x = \int r^2\, dm = \int_0^{2\pi} (R\, \mathrm{Sin}[\theta])^2\, \rho\, R\, d\theta = \frac{1}{2} (2\pi R \rho) R^2 = \frac{1}{2} M R^2 \tag{22}$$

Alternatively, we could have used the Perpendicular Axis Theorem, since the ring is spherically symmetric. Having found that $I_z = M R^2$ in the last example and using the symmetry of the problem to determine $I_x = I_y$, the Perpendicular Axis Theorem states that $I_x + I_y = I_z$ and therefore $I_x = \frac{1}{2} I_z = \frac{1}{2} M R^2$. □

Example

Calculate the moment of inertia of a rectangle of mass M with side lengths a and b about an axis going through its center perpendicular to the plane.



Solution

Defining the mass density $\rho = \frac{M}{a\, b}$,

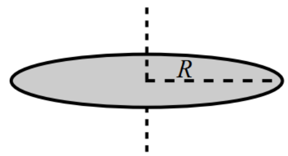
$$I = \int r^2\, dm = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} (x^2 + y^2)\, \rho\, dx\, dy = \frac{1}{12} a\, b\, (a^2 + b^2)\, \rho = \frac{1}{12} M (a^2 + b^2) \tag{23}$$

Alternatively, we could have used the Perpendicular Axis Theorem, since (using the result of the thin rod)

$I_x = \frac{1}{12} M\, b^2$ and $I_y = \frac{1}{12} M\, a^2$ so that $I_z = I_x + I_y = \frac{1}{12} M\left(a^2 + b^2\right)$. □

Example

Calculate the moment of inertia of a disk of mass M and radius R about the axis going through the ring's center perpendicular to the plane of the disk.



Solution

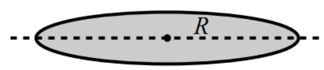
Using polar coordinates and defining the mass density $\rho = \frac{M}{\pi R^2}$,

$$I = \int r^2\, dm = \int_0^{2\pi} \int_0^R r^2 \rho\, r\, dr\, d\theta = \frac{\pi}{2} \rho\, R^4 = \frac{1}{2} M\, R^2 \tag{24}$$

The $\frac{1}{2}$ factor is very interesting, and it tells you a neat property about circles and how their areas are distributed relative to the origin! □

Example

Calculate the moment of inertia of a disk of mass M and radius R about the axis going through the ring's center parallel to the plane of the disk.



Solution

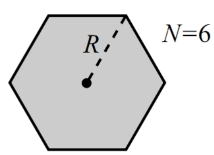
Using polar coordinates and defining the mass density $\rho = \frac{M}{\pi R^2}$,

$$I = \int r^2\, dm = \int_0^{2\pi} \int_0^R r^2\, \mathrm{Sin}[\theta]^2 \rho\, r\, dr\, d\theta = \frac{\pi}{4} \rho\, R^4 = \frac{1}{4} M\, R^2 \tag{25}$$

By symmetry, this also agrees with the Perpendicular Axis Theorem. □

Example

Calculate the moment of inertia of a regular N-gon of mass M and "radius" R through the axis going through its center and perpendicular to the plane.



Solution

The N-gon equals N isosceles triangles put together. Therefore its moment of inertia will equal the moment of inertia of one of these isosceles triangles multiplied by N. Orient one such isosceles triangle so that its edges lies on $\theta = \frac{\pi}{2} - \frac{\pi}{N}$, $\theta = \frac{\pi}{2} + \frac{\pi}{N}$, and $y = R\, \mathrm{Cos}\left[\frac{\pi}{N}\right]$ (for example, for $N = 6$ this is the triangle whose right edge is dashed in the figure above. For a general N, this may require you to rotate the figure). Substituting $y = r\, \mathrm{Sin}[\theta]$ into this last relation, we can compute the moment of inertia of this triangle as

$$
\begin{aligned}
I &= \int r^2 \, dm \\
&= N \int_{\frac{\pi}{2}-\frac{\pi}{N}}^{\frac{\pi}{2}+\frac{\pi}{N}} \int_0^{R \frac{\mathrm{Cos}\left[\frac{\pi}{N}\right]}{\mathrm{Sin}[\theta]}} r^2 \, \rho \, r \, dr \, d\theta \\
&= N \rho \int_{\frac{\pi}{2}-\frac{\pi}{N}}^{\frac{\pi}{2}+\frac{\pi}{N}} \frac{1}{4} \left(R \frac{\mathrm{Cos}\left[\frac{\pi}{N}\right]}{\mathrm{Sin}[\theta]}\right)^4 d\theta \\
&= \frac{1}{24} N R^4 \rho \left(4 \, \mathrm{Sin}\left[\frac{2\pi}{N}\right] + \mathrm{Sin}\left[\frac{4\pi}{N}\right]\right)
\end{aligned}
$$

where we have used *Mathematica* to compute and simplify this last integral. We now need to compute ρ by dividing M by the area of the N-gon. The area of the isosceles triangle we have been analyzing equals $\left(R \, \mathrm{Sin}\left[\frac{\pi}{N}\right]\right)\left(R \, \mathrm{Cos}\left[\frac{\pi}{N}\right]\right)$ so the area of the N-gon equals $\frac{1}{2} N R^2 \, \mathrm{Sin}\left[\frac{2\pi}{N}\right]$ so that $\rho = \frac{2M}{N R^2 \, \mathrm{Sin}\left[\frac{2\pi}{N}\right]}$. Substituting this in,

$$
\begin{aligned}
I &= \frac{1}{24} N R^4 \rho \left(4 \, \mathrm{Sin}\left[\frac{2\pi}{N}\right] + \mathrm{Sin}\left[\frac{4\pi}{N}\right]\right) \\
&= \frac{1}{6} M R^2 \left(2 + \mathrm{Cos}\left[\frac{2\pi}{N}\right]\right)
\end{aligned} \tag{27}
$$

Here are some values in the form $\left(N, \frac{I}{M R^2}\right)$: $\left(3, \frac{1}{4}\right)$, $\left(4, \frac{1}{3}\right)$, $\left(5, \frac{1}{12}\right)$, $\left(\infty, \frac{1}{2}\right)$. Note that these particular values form an arithmetic progression. □

Calculating Moments of Inertia - Pro Version

For some objects with certain symmetries, it is possible to calculate the moment of inertia I without doing any integrals. All that is needed is a scaling argument and the Parallel Axis Theorem.

Let's compare the moments of inertia I_{small} of a thin rod of mass M and length L with the moment of inertia I_{large} for a stick of mass $2M$ and length $2L$. Defining the mass per unit length $\rho = \frac{M}{L}$, we have

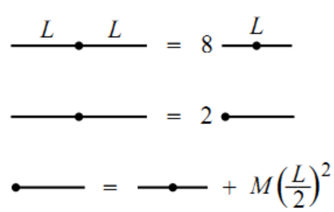
$$I_{\text{small}} = \int_{-L/2}^{L/2} x^2 \, \rho \, dx \tag{28}$$

Change variables to $y = 2x$, we obtain

$$I_{\text{small}} = \int_{-L}^{L} \left(\frac{y}{2}\right)^2 \rho \, \frac{dy}{2} = \frac{1}{8} \int_{-L}^{L} y^2 \, \rho \, dy \tag{29}$$

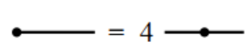
But since this last integral is $I_{\text{large}} = \int_{-L}^{L} y^2 \, \rho \, dy$, we see that $I_{\text{large}} = 8 \, I_{\text{small}}$.

Using this idea, we can cleverly find the moment of inertia I_{small} using the Parallel Axis Theorem. The technique is most easily illustrated with pictures. If we denote the moment of inertia of an object by a picture of the object, with a dot signifying the axis, then we have



(30)

The first line comes from the scaling argument, the second line comes from the fact that moments of inertia simply add (the left-hand side is two copies of the right-hand side, attached at the pivot), and the third line comes from the parallel-axis theorem. Equating the right-hand sides of the first two equations gives



(31)

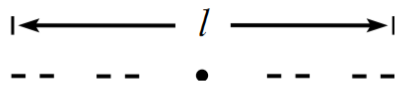
Plugging this expression for into the third equation gives the desired result,

 $= \frac{1}{12} M L^2$ (32)

This trick lets you easily calculate the moment of inertia of fractal objects!

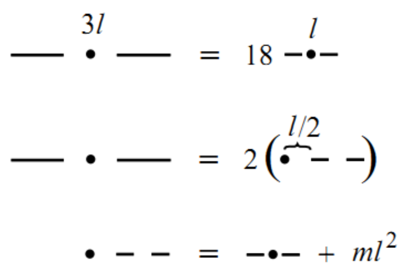
Example

Take a stick of length l, and remove the middle third. Then remove the middle third from each of the remaining two pieces. Then remove the middle third from each of the remaining four pieces, and so on, forever. Let the final object have mass m. Calculate the moment of inertia about an axis through its center, perpendicular to stick. (Hint: Be careful about how the mass scales.)



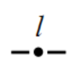
Solution

Our object is self-similar to an object 3 times as big, so let's increase the length by a factor of 3 and see what happens to I. In the integral $\int x^2\, dm$, each x pick up a factor of 3, yielding a factor of 9 from x^2. But what happens to the dm? Tripling the size of our object increases its mass by a factor of 2, because the new object is simply made up of two of the smaller ones, plus some empty space in the middle. So the dm picks up a factor of 2. Therefore, the I for an object of length $3\,l$ is 18 times the I for an object of length l, assuming that the axes pass through any two corresponding points. With pictures, we have



(33)

Therefore, we can solve this to obtain

 $= \frac{1}{8} m l^2$ (34)

When we increase the length of our object by a factor of 3 here, the factor of 2 in the dm is larger than the factor of 1 relevant to a zero-dimensional object, but smaller than the factor of 3 relevant to a one-dimensional object. So in some sense our object has a dimension between 0 and 1. It is reasonable to define the dimension, d, of an object as the number for which r^d is the increase in "volume" when the dimensions are increased by a factor of r. In this problem, we have $3^d = 2$ so that $d = \mathrm{Log}_3[2] \approx 0.63$ □